

# Uniqueness in inverse boundary value problems for fractional diffusion equations

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## Abstract

We consider an inverse boundary value problem for diffusion equations with multiple fractional time derivatives. We prove the uniqueness in determining a number of fractional time-derivative terms, the orders of the derivatives and spatially varying coefficients.

**Keywords:** fractional diffusion equation, inverse problem, determination of fractional orders, Dirichlet-to-Neumann map, multi-time-fractional derivatives.

## 1 Introduction

Let  $\Omega$  be an open bounded domain in  $\mathbb{R}^d$  with smooth boundary, for example, of  $C^2$ -class,  $\nu$  be the outward unit normal vector to  $\partial\Omega$ . We denote  $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ . Let  $T > 0$  be fixed arbitrarily. Consider the following initial-boundary value problem

$$\begin{cases} \sum_{j=1}^{\ell} p_j(x) \partial_t^{\alpha_j} u(x, t) = \Delta u(x, t) + p(x)u(x, t), & (x, t) \in \Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = \lambda(t)g(x), & 0 < t < T, \end{cases} \quad (1)$$

where  $\alpha_j$ ,  $j = 1, \dots, \ell$ , are positive constants such that

$$0 < \alpha_{\ell} < \dots < \alpha_1 < 1. \quad (2)$$

Here and henceforth, for  $\alpha \in (0, 1)$ , by  $\partial_t^{\alpha} v$  we denote the Caputo fractional derivative with respect to  $t$ :

$$\partial_t^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} v(\tau) d\tau \quad (3)$$

(e.g., Podlubny [23]) and  $\Gamma$  is the Gamma function.

The case of  $\ell = 1$ , i.e., a single-term time-fractional diffusion equation is used for example as a model equation for the anomalous diffusion phenomena in heterogeneous media (e.g., Metzler and Klafter [21]). We further refer to Kilbas, Srivastava and Trujillo [11], Luchko [15], Luchko and Gorenflo [18], Mainardi [19], Podlubny [23], Sakamoto and Yamamoto [24].

On the other hand, diffusion equations whose orders of the derivatives change in time and/or spatial coordinates, are proposed as feasible models (e.g., Chechkin, Gorenflo and Sokolov [3], Coimbra [5], Lorenzo and Hartley [14], Mainardi, Mura, Pagnini and Gorenflo [20], Pedro, Kobayshi, Pereira and Coimbra [22],

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Sokolov, Chechkin and Klafter [25].) Among them, we consider a multi-term time-fractional diffusion equation (1).

For applying (1) as model equation, in order to interpret measurement data, we usually need to suitably choose  $\ell, p_j, \alpha_j, p$  which describe physical properties of the diffusion process under consideration. This is our inverse problem, and we discuss the uniqueness as the fundamental theoretical topic for the inverse problem.

Henceforth, for  $\ell \in \mathbb{N}$ , we set  $\vec{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in (0, 1)^\ell$  where  $\alpha_\ell < \alpha_{\ell-1} < \dots < \alpha_1$ . We note that also  $\ell$  is unknown parameter in the inverse problem.

We state

**Inverse problem** Let  $\lambda \neq 0$  be fixed. For  $g \in H^{\frac{3}{2}}(\partial\Omega)$ , we define the Dirichlet-to-Neumann map by

$$\Lambda(\ell, \vec{\alpha}, p_j, p)g := \frac{\partial u}{\partial \nu}|_{\partial\Omega \times (0, T)} \in L^2(0, T; H^{\frac{1}{2}}(\partial\Omega)).$$

Can we uniquely determine  $(\ell, \vec{\alpha}, p_j, p)$  by the map  $\Lambda(\ell, \vec{\alpha}, p_j, p) : H^{\frac{3}{2}}(\partial\Omega) \rightarrow L^2(0, T; H^{\frac{1}{2}}(\partial\Omega))$ ?

In Section 2, we prove that the Dirichlet-to-Neumann map is well defined. Our inverse problem is based on the Dirichlet-to-Neumann map, and for elliptic equations, there have been numerous important works. Here we do not intend any lists of references and we refer only to Imanuvilov and Yamamoto [9], Isakov [10], Sylvester and Uhlmann [26] and the references therein.

For the statement of our main results, we introduce some notations. As an admissible set of unknown fractional orders including numbers and coefficients, we set

$$\mathcal{U} = \{(\ell, \vec{\alpha}, p_1, \dots, p_\ell, p) \in \mathbb{N} \times (0, 1)^\ell \times C^\infty(\overline{\Omega})^{\ell+1}; p_j \geq 0, \neq 0, j = 2, 3, \dots, \ell, p_1 > 0, p \leq 0 \text{ on } \overline{\Omega}\}.$$

where  $\vec{\alpha} := (\alpha_1, \dots, \alpha_\ell)$  such that  $\alpha_\ell < \alpha_{\ell-1} < \dots < \alpha_1$ . For  $\theta \in (0, \frac{\pi}{2})$ , we further set

$$\Omega_\theta := \{z \in \mathbb{C}; z \neq 0, |\arg z| < \theta\}.$$

We are ready to state our main result.

**Theorem 1.1** (Uniqueness). *Let  $(\ell, \vec{\alpha}, p_j, p) \in \mathcal{U}$  and  $(m, \vec{\beta}, q_j, q) \in \mathcal{U}$ . Assume that for some  $\theta \in (0, \frac{\pi}{2})$  the function  $\lambda \neq 0$  can be analytically extended to  $\Omega_\theta$  with  $\lambda(0) = 0$  and  $\lambda'(0) = 0$  and there exists a constant  $C_0 > 0$  such that  $|\lambda^{(k)}(t)| \leq C_0 e^{C_0 t}$ ,  $t > 0$ ,  $0 \leq k \leq 2$ . Then  $\ell = m$ ,  $\vec{\alpha} = \vec{\beta}$ ,  $p_j = q_j$ ,  $1 \leq j \leq \ell$  and  $p = q$  provided*

$$\Lambda(\ell, \vec{\alpha}, p_j, p)g = \Lambda(m, \vec{\beta}, q_j, q)g, \quad g \in H^{\frac{3}{2}}(\partial\Omega). \quad (4)$$

The assumption  $p \leq 0$  on  $\overline{\Omega}$  is necessary for proving that  $|u(x, t)| = O(e^{C_1 t})$  as  $t \rightarrow \infty$  with some constant  $C_1 > 0$ . Such an estimate is sufficient for taking the Laplace transforms of  $u$ , which is a key of the proof of Theorem 1.1. In this paper, we do not discuss the inverse problem without the condition  $p \leq 0$ .

In  $\mathcal{U}$ , we can relax the regularity of  $p, p_1, \dots, p_\ell$  but we do not discuss here. Moreover, in the two dimensional case of  $d = 2$ , thanks to Imanuvilov and Yamamoto [8], we can prove a sharp uniqueness result where Dirichlet inputs and Neumann outputs can be restricted an arbitrary subboundary and the required regularity of unknown coefficients is relaxed.

**Corollary 1.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with smooth boundary  $\partial\Omega$  and  $\Gamma \subset \partial\Omega$  be an arbitrarily given subboundary and let  $\gamma > 2$  be arbitrarily fixed. We assume the  $\lambda$  satisfies the same conditions as in Theorem 1.1. We set*

$$\widehat{\mathcal{U}} = \{(\ell, \vec{\alpha}, p_1, \dots, p_\ell, p) \in \mathbb{N} \times (0, 1)^\ell \times W^{2, \infty}(\Omega) \times (W^{1, \gamma}(\Omega))^\ell; p_j \geq 0, p_j \neq 0, j = 2, 3, \dots, \ell, p_1 > 0, p \leq 0 \text{ on } \overline{\Omega}\}.$$

If  $(\ell, \vec{\alpha}, p_j, p), (m, \vec{\beta}, q_j, q) \in \widehat{\mathcal{U}}$  satisfy

$$\Lambda(\ell, \vec{\alpha}, p_j, p)g = \Lambda(m, \vec{\beta}, q_j, q)g \quad \text{on } \Gamma$$

for all  $g \in H^{\frac{3}{2}}(\partial\Omega)$  with  $\text{supp } g \subset \Gamma$ , then  $\ell = m$ ,  $\vec{\alpha} = \vec{\beta}$ ,  $p_j = q_j$ ,  $1 \leq j \leq \ell$  and  $p = q$ .

As for single-term time-fractional diffusion equations, there are not many works on inverse problems in spite of the physical and practical importance and see e.g., Cheng, Nakagawa, Yamamoto and Yamazaki [4], Li, Zhang, Jia and Yamamoto [6], Hatano, Nakagawa, Wang and Yamamoto [7]. Moreover for inverse problems for multi-term time-fractional diffusion equations, to the best knowledge of the authors, there are no existing results.

The rest of the paper is organized as follows. In Section 2, we prove properties of solutions to (1) which are necessary for the proof of Theorem 1.1. In particular, the  $t$ -analyticity of solution is essential. In Section 3, by applying the Laplace transforms of the solutions to (1) and reducing our inverse problem to the inverse boundary value problem for elliptic equations, we complete the proof of Theorem 1.1.

## 2 Forward problem

For  $\theta \in (0, \frac{\pi}{2})$  and  $T > 0$ , we set

$$\Omega_\theta := \{z \in \mathbb{C}; z \neq 0, |\arg z| < \theta\}, \quad \Omega_{\theta,T} := \{z \in \Omega_\theta; |z| < T\}.$$

In this section, we establish the analyticity of the solution  $u$  to the initial-boundary value problem (1) as well as the unique existence of the solution. As for other results for solutions to (1), see Beckers and Yamamoto [2], Li and Yamamoto [12], Li, Liu and Yamamoto [13], Luchko [16], [17] for example.

**Theorem 2.1.** *Let  $(\ell, \vec{\alpha}, p_j, p) \in \mathcal{U}$  and  $T > 0$  be arbitrarily given. Assume that  $g \in H^{\frac{3}{2}}(\partial\Omega)$ ,  $\lambda(0) = 0$ ,  $\lambda'(0) = 0$ , for  $\theta \in (0, \frac{\pi}{2})$  and  $T > 0$ , the function  $\lambda(t)$  can be analytically extended to  $\Omega_\theta$  and  $\lambda \in W^{2,\infty}(\Omega_{\theta,T})$ . Then there exists a unique mild solution  $u \in C([0, \infty); H^2(\Omega))$  and  $Au(t): (0, \infty) \rightarrow H^2(\Omega)$  can be analytically extended to  $\Omega_\theta$ .*

Moreover, for  $g \in C^\infty(\partial\Omega)$  and any  $T > 0$ , we have

$$\|u\|_{C(\bar{\Omega} \times [0, T])} \leq \|g\|_{C(\partial\Omega)} \|\lambda\|_{C[0, T]}. \quad (5)$$

*Proof.* The proof is based on the following observation. By the Sobolev extension theorem,  $g \in H^{\frac{3}{2}}(\partial\Omega)$  allows us to choose  $\tilde{g} \in H^2(\Omega)$  such that  $\tilde{g}|_{\partial\Omega} = g$ . Introducing the new unknown function  $\tilde{u}(x, t) = u(x, t) - \lambda(t)\tilde{g}(x)$ , we can rewrite (1) as

$$\begin{cases} \partial_t^{\alpha_1} \tilde{u} + \sum_{j=2}^{\ell} \tilde{p}_j(x) \partial_t^{\alpha_j} \tilde{u} = \operatorname{div}(\frac{1}{p_1(x)} \nabla \tilde{u}) + B(x) \cdot \nabla \tilde{u} + b(x) \tilde{u} + F(x, t), & \text{in } \Omega \times (0, T), \\ \tilde{u}(x, 0) = 0, & x \in \Omega, \\ \tilde{u}|_{\partial\Omega} = 0, & 0 < t < T, \end{cases} \quad (6)$$

where  $\tilde{p}_j(x) := \frac{p_j(x)}{p_1(x)}$ ,  $j = 2, \dots, \ell$ ,  $B(x) := \nabla(\frac{-1}{p_1(x)})$ ,  $b(x) := \frac{p(x)}{p_1(x)}$  and

$$F(x, t) := \frac{1}{p_1(x)} \left( \lambda(t) \Delta \tilde{g}(x) + \lambda(t) p(x) \tilde{g}(x) - \sum_{j=1}^{\ell} (\partial_t^{\alpha_j} \lambda)(t) p_j(x) \tilde{g}(x) \right). \quad (7)$$

Then  $F(x, \cdot)$  can be analytically extended to  $\Omega_\theta$ . In fact, it is sufficient to prove that  $\partial_t^\alpha \lambda$  can be analytically extended to  $\Omega_{\theta,T}$  with any  $\alpha \in (0, 1)$  and  $T > 0$ . Let  $z \in \Omega_{\theta,T}$  be arbitrarily fixed. We set

$$\lambda_\alpha(z) := \frac{z^{-\alpha-1}}{\Gamma(1-\alpha)} \int_0^1 (1-\eta)^{-\alpha} \frac{\partial(\lambda(\eta z))}{\partial \eta} d\eta = \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-s)^{-\alpha} \frac{d\lambda(s)}{ds} ds.$$

Here the integral is considered on the segment from 0 to  $z$  in  $\mathbb{C}$ . Then we can see that  $\lambda_\alpha(t) = \partial_t^\alpha \lambda(t)$  for  $t > 0$ . For any small  $\varepsilon > 0$ , we set

$$\lambda_\alpha^\varepsilon(z) = \frac{z^{-\alpha-1}}{\Gamma(1-\alpha)} \int_0^{1-\varepsilon} (1-\eta)^{-\alpha} \frac{\partial(\lambda(\eta z))}{\partial \eta} d\eta.$$

By the analyticity of  $\lambda$  in  $\Omega_\theta$ , we see that  $\lambda_\alpha^\varepsilon$  is analytic in  $\Omega_{\theta,T}$ . Let  $\varepsilon_0 > 0$  be an arbitrarily fixed small constant. For any  $z \in \Omega_{\theta, \varepsilon_0, T} := \{z \in \Omega_{\theta,T}; |z| > \varepsilon_0\}$ , we have

$$|\lambda_\alpha^\varepsilon(z) - \lambda_\alpha(z)| \leq \frac{\varepsilon_0^{-\alpha-1}}{\Gamma(1-\alpha)} \int_{1-\varepsilon}^1 (1-\eta)^{-\alpha} \left| \frac{\partial(\lambda(\eta z))}{\partial \eta} \right| d\eta \leq \frac{\varepsilon_0^{-\alpha-1}}{\Gamma(1-\alpha)} \sup_{0 < \eta < 1} \left| \frac{\partial(\lambda(\eta z))}{\partial \eta} \right| \int_{1-\varepsilon}^1 (1-\eta)^{-\alpha} d\eta.$$

Therefore, for any fixed  $\varepsilon_0 > 0$  and  $T > 0$ , we see that

$$\sup_{z \in \Omega_{\theta, \varepsilon_0, T}} |\lambda_{\alpha}^{\varepsilon}(z) - \lambda_{\alpha}(z)| \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Since  $\lambda_{\alpha}^{\varepsilon}$  is analytic in  $\Omega_{\theta, T}$ , we see that  $\lambda_{\alpha}$  is analytic in  $\Omega_{\theta, T}$ , because  $\lambda_{\alpha}$  is the uniform convergent limit of analytic  $\lambda_{\alpha}^{\varepsilon}$  in any compact subset of  $\Omega_{\theta, T}$ . Thus we completed the proof that  $F(\cdot, t)$  can be analytically extended to  $\Omega_{\theta}$ .

Next we estimate  $F$ . Let  $T \geq 1$ . First we have

$$\|F\|_{L^{\infty}(0, T; L^2(\Omega))} \leq C \left( \|\lambda\|_{L^{\infty}(0, T)} + \sum_{j=1}^{\ell} \|\partial_t^{\alpha_j} \lambda\|_{L^{\infty}(0, T)} \right).$$

Here and henceforth  $C > 0$  denotes a generic constant which is independent of  $T, t > 0, z \in \Omega_{\theta}$ , but dependent on  $d, \Omega, g, \theta, p, p_1, \dots, p_{\ell}, \alpha_1, \dots, \alpha_{\ell}$ . We have

$$|\partial_t^{\alpha_j} \lambda(t)| = \frac{1}{\Gamma(1 - \alpha_j)} \left| \int_0^t (t-s)^{-\alpha_j} \frac{d\lambda}{ds}(s) ds \right| \leq C \int_0^t (t-s)^{-\alpha_j} ds \|\lambda\|_{C^1[0, T]} \leq CT \|\lambda\|_{C^1[0, T]},$$

and so

$$\|F\|_{L^{\infty}(0, T; L^2(\Omega))} \leq CT \|\lambda\|_{C^1[0, T]}.$$

Moreover, by  $0 < \alpha_j < 1$ ,  $\lambda'(0) = 0$  and integration by parts yield

$$\begin{aligned} \partial_t^{\alpha_j} \lambda(t) &= \frac{1}{\Gamma(1 - \alpha_j)} \int_0^t (t-s)^{-\alpha_j} \frac{d\lambda}{ds}(s) ds \\ &= \frac{1}{\Gamma(1 - \alpha_j)} \left( \left[ \lambda'(s) \frac{(t-s)^{1-\alpha_j}}{1-\alpha_j} \right]_{s=t}^{s=0} + \int_0^t \frac{(t-s)^{1-\alpha_j}}{1-\alpha_j} \frac{d^2\lambda}{ds^2}(s) ds \right) \\ &= \frac{1}{\Gamma(1 - \alpha_j)} \int_0^t \frac{(t-s)^{1-\alpha_j}}{1-\alpha_j} \frac{d^2\lambda}{ds^2}(s) ds. \end{aligned}$$

Therefore

$$\partial_t \partial_t^{\alpha_j} \lambda(t) = \frac{1}{\Gamma(1 - \alpha_j)} \int_0^t (t-s)^{-\alpha_j} \frac{d^2\lambda}{ds^2}(s) ds,$$

and so

$$\|\partial_t \partial_t^{\alpha_j} \lambda\|_{L^{\infty}(0, T)} \leq C \int_0^t (t-s)^{-\alpha_j} ds \|\lambda\|_{C^2[0, T]} \leq CT \|\lambda\|_{C^2[0, T]}.$$

Hence  $\|\partial_t F\|_{L^{\infty}(0, T; L^2(\Omega))} \leq CT \|\lambda\|_{C^2[0, T]}$ . Consequently

$$\|F\|_{W^{1, \infty}(0, T; L^2(\Omega))} \leq CT \|\lambda\|_{C^2[0, T]} \quad (8)$$

for all  $T \geq 1$ . Next we estimate  $\|F(\cdot, z)\|_{L^2(\Omega)}$  for  $z \in \Omega_{\theta, T}$ . Noting that  $\tilde{\lambda}(\eta) = \lambda(\eta z)$  and  $\frac{d\tilde{\lambda}}{d\eta}(\eta) = z \frac{d\lambda}{d\eta}(\eta z)$  for  $0 < \eta < 1$  and  $z \in \Omega_{\theta, T}$ , and

$$\lambda_{\alpha_j}(z) := \frac{1}{\Gamma(1 - \alpha_j)} \int_0^z (z-s)^{-\alpha_j} \frac{d\lambda}{ds}(s) ds = \frac{z^{-\alpha_j}}{\Gamma(1 - \alpha_j)} \int_0^1 (1-\eta)^{-\alpha_j} \frac{d\tilde{\lambda}}{d\eta}(\eta) d\eta$$

we have

$$\begin{aligned} \|\lambda_{\alpha_j}\|_{L^{\infty}(\Omega_{\theta, T})} &\leq C |z|^{-\alpha_j} \int_0^1 (1-\eta)^{-\alpha_j} d\eta |z| \sup_{s \in [0, z]} \left| \frac{d\lambda}{ds}(s) \right| \\ &\leq C |z|^{1-\alpha_j} \|\lambda\|_{W^{1, \infty}(\Omega_{\theta, T})} \leq CT \|\lambda\|_{W^{1, \infty}(\Omega_{\theta, T})}. \end{aligned}$$

Here  $[0, z]$  denotes the closed segment in  $\mathbb{C}$  from 0 to  $z$ . Arguing similarly to the case of  $t \in [0, T]$ , we obtain

$$\|F\|_{W^{1, \infty}(\Omega_{\theta, T}; L^2(\Omega))} \leq CT \|\lambda\|_{W^{2, \infty}(\Omega_{\theta, T})}. \quad (9)$$

Next we define operator  $A$  in  $H^2(\Omega) \cap H_0^1(\Omega)$  to be

$$(A\psi)(x) = -\operatorname{div}\left(\frac{1}{p_1(x)}\nabla\psi(x)\right), \quad x \in \Omega, \quad \psi \in H^2(\Omega) \cap H_0^1(\Omega).$$

Here and henceforth  $\{\lambda_k, \phi_k\}_{k=1}^\infty$  denotes the eigensystem of the elliptic operator  $A$  such that  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$ ,  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ ,  $A\phi_k = \lambda_k \phi_k$  and  $\{\phi_k\} \subset H^2(\Omega) \cap H_0^1(\Omega)$  forms an orthonormal basis of  $L^2(\Omega)$ . Then we can define the fractional power  $A^\gamma$  for  $\gamma > 0$  of the operator  $A$  (e.g., Tanabe [27]), and we see that

$$D(A^\gamma) = \left\{ \psi \in L^2(\Omega) : \sum_{n=1}^\infty \lambda_n^{2\gamma} |(\psi, \phi_n)_{L^2(\Omega)}|^2 < \infty \right\}$$

is a Hilbert space with the norm

$$\|\psi\|_{D(A^\gamma)} = \left( \sum_{n=1}^\infty \lambda_n^{2\gamma} |(\psi, \phi_n)_{L^2(\Omega)}|^2 \right)^{\frac{1}{2}}.$$

We further define the operator  $S(t) : L^2(\Omega) \rightarrow L^2(\Omega)$  for  $t > 0$  by

$$S(t)a := \sum_{n=1}^\infty (a, \phi_n)_{L^2(\Omega)} E_{\alpha,1}(-\lambda_n t^{\alpha_1}) \phi_n \text{ in } L^2(\Omega) \quad (10)$$

for  $a \in L^2(\Omega)$ , where  $E_{\alpha,\beta}(z)$  is Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \quad \alpha > 0, \quad \beta \in \mathbb{R}.$$

The above formula and the classical asymptotics

$$\Gamma(\eta) = e^{-\eta} \eta^{\eta-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left( 1 + O\left(\frac{1}{\eta}\right) \right) \quad \text{as } \eta \rightarrow +\infty, \quad \eta > 0 \quad (11)$$

(e.g., Abramowitz and Stegun [1], p.257) imply that the radius of convergence is  $\infty$  and so  $E_{\alpha,\beta}(z)$  is an entire function of  $z \in \mathbb{C}$ .

Moreover the term-wise differentiations are possible and give

$$\begin{aligned} S'(t)a &:= - \sum_{n=1}^\infty \lambda_n (a, \phi_n)_{L^2(\Omega)} t^{\alpha_1-1} E_{\alpha,1}(-\lambda_n t^{\alpha_1}) \phi_n \text{ in } L^2(\Omega) \\ S''(t)a &:= - \sum_{n=1}^\infty \lambda_n (a, \phi_n)_{L^2(\Omega)} t^{\alpha_1-2} E_{\alpha,1}(-\lambda_n t^{\alpha_1}) \phi_n \text{ in } L^2(\Omega) \end{aligned}$$

for  $a \in L^2(\Omega)$ ,  $t > 0$  (e.g., Podlubny [23]).

From the definition of (10) and the property of Mittag-Leffler function,  $S'(z)$  and  $S''(z)$  are analytic in the sector  $\Omega_\theta$  and by Theorem 1.6 in [23] (p.35), we can prove that there exists a constant  $C > 0$ , which is independent of  $z$  such that

$$\|S(z)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C, \quad z \in \Omega_\theta, \quad (12)$$

$$\|A^{\gamma-1} S'(z)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C |z|^{\alpha_1-1-\alpha_1\gamma}, \quad z \in \Omega_\theta, \quad 0 \leq \gamma \leq 1, \quad (13)$$

$$\|A^{\gamma-1} S''(z)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C |z|^{\alpha_1-2-\alpha_1\gamma}, \quad z \in \Omega_\theta, \quad 0 \leq \gamma \leq 1. \quad (14)$$

We verify only (12), because the proofs of (13) and (14) are similar. In fact, an estimate of  $E_{\alpha,1}(-t)$  (e.g., Theorem 1.6 in [23], p.35) implies that (10) holds for  $t \in \Omega_\theta$  and we have

$$\begin{aligned} \|S(z)a\|_{L^2(\Omega)}^2 &= \sum_{n=1}^\infty (a, \phi_n)_{L^2(\Omega)}^2 |E_{\alpha,1}(-\lambda_n z^{\alpha_1})|^2 \\ &\leq C \sum_{n=1}^\infty (a, \phi_n)_{L^2(\Omega)}^2 \frac{1}{1 + |\lambda_n z^{\alpha_1}|} \leq C \sum_{n=1}^\infty (a, \phi_n)_{L^2(\Omega)}^2 = C \|a\|_{L^2(\Omega)}^2, \end{aligned}$$

which proves (12).

In view of  $1 > \alpha_1 > \dots > \alpha_\ell > 0$ , regarding  $-\sum_{j=2}^\ell \tilde{p}_j \partial_t^{\alpha_j} \tilde{u} + B \cdot \nabla \tilde{u} + b\tilde{u}$  as non-homogeneous term in (6), and we have

$$\begin{aligned} \tilde{u}(t) &= - \int_0^t A^{-1} S'(t-s) (B \cdot \nabla \tilde{u}(s) + b\tilde{u}(s) + F(s)) ds \\ &\quad + \sum_{j=2}^\ell \int_0^t A^{-1} S'(t-s) \tilde{p}_j \partial_t^{\alpha_j} \tilde{u}(s) ds. \end{aligned}$$

Now we calculate the right-hand side. Noting by the definition of Caputo fractional derivative and the Fubini theorem, similarly to [2] or [24], we change the orders of integrations to derive

$$\begin{aligned} \int_0^t A^{-1} S'(t-s) \left( \tilde{p}_j \partial_t^{\alpha_j} \tilde{u}(s) \right) ds &= \int_0^t A^{-1} S'(t-s) \frac{1}{\Gamma(1-\alpha_j)} \left( \int_0^s (s-r)^{-\alpha_j} \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr \right) ds \\ &= \int_0^t \left( \int_r^t A^{-1} S'(t-s) \frac{(s-r)^{-\alpha_j}}{\Gamma(1-\alpha_j)} ds \right) \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr \\ &= \int_0^t \left( \int_0^{t-r} A^{-1} S'(t-r-\xi) \frac{\xi^{-\alpha_j}}{\Gamma(1-\alpha_j)} d\xi \right) \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr. \end{aligned}$$

Here in the last equality, we used the change of variable  $\xi := s - r$ . Decomposing the integrand, we obtain

$$\begin{aligned} &\int_0^t \left( \int_0^{t-r} A^{-1} S'(t-r-\xi) \xi^{-\alpha_j} d\xi \right) \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr \\ &= \int_0^t \left( \int_0^{t-r} A^{-1} S'(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) d\xi \right) \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr \\ &\quad + \int_0^t \left( \int_0^{t-r} A^{-1} S'(t-r-\xi) d\xi \right) (t-r)^{-\alpha_j} \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr =: I_1(t) + I_2(t). \end{aligned}$$

Here we should understand that

$$I_1(t) = \lim_{\varepsilon_1, \varepsilon_2 \downarrow 0} \int_0^t \left( \int_{\varepsilon_2}^{t-r-\varepsilon_1} A^{-1} S'(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) d\xi \right) \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr,$$

but throughout the following calculations, we can prove that the resulting integrals are all convergent, so that we present the calculations without such passage to limits.

Integration by parts yields

$$\begin{aligned} I_1(t) &= \int_0^{t-r} A^{-1} S'(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) \tilde{p}_j \tilde{u}(r) d\xi \Big|_{r=0}^{r=t} \\ &\quad + \int_0^t \left( \int_0^{t-r} A^{-1} S''(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) d\xi \right) \tilde{p}_j \tilde{u}(r) dr \\ &\quad + \alpha_j \int_0^t \left( \int_0^{t-r} A^{-1} S'(t-r-\xi) d\xi \right) (t-r)^{-\alpha_j-1} \tilde{p}_j \tilde{u}(r) dr \\ &\quad + \int_0^t \lim_{\xi \rightarrow t-r} A^{-1} S'(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) \tilde{p}_j \tilde{u}(r) dr. \end{aligned}$$

From (13) and the fact  $\alpha_1 > \alpha_j$  for  $j = 2, \dots, \ell$ , we deduce

$$\begin{aligned}
& \left\| \int_0^{t-r} A^{-1} S'(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) d\xi \right\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \\
& \leq \left\| \int_0^{t-r} A^{-1} S'(t-r-\xi) \xi^{-\alpha_j} d\xi \right\|_{L^2(\Omega) \rightarrow L^2(\Omega)} + \left\| \int_0^{t-r} A^{-1} S'(t-r-\xi) (t-r)^{-\alpha_j} d\xi \right\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \\
& \leq C \int_0^{t-r} (t-r-\xi)^{\alpha_1-1} \xi^{(1-\alpha_j)-1} d\xi + C \int_0^{t-r} (t-r-\xi)^{\alpha_1-1} d\xi (t-r)^{-\alpha_j} \\
& \leq C(t-r)^{\alpha_1-\alpha_j} \frac{\Gamma(\alpha_1)\Gamma(1-\alpha_j)}{\Gamma(\alpha_1+1-\alpha_j)} + \frac{C}{\alpha_1} (t-r)^{\alpha_1-\alpha_j} \rightarrow 0 \quad \text{as } r \rightarrow t.
\end{aligned}$$

Moreover, by (13) and  $|\xi^{-\alpha_j} - (t-r)^{-\alpha_j}| \leq C\xi^{-\alpha_j-1}|t-r-\xi|$  for  $0 < \xi < t-r$ , we have

$$\|A^{-1} S'(t-r-\xi)(\xi^{-\alpha_j} - (t-r)^{-\alpha_j})\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C(t-r-\xi)^{\alpha_1} \xi^{-\alpha_j-1} \rightarrow 0 \text{ as } \xi \rightarrow t-r$$

for  $t > r$ . Therefore, by  $\tilde{u}(0) = 0$ , we obtain

$$\begin{aligned}
I_1(t) &= \int_0^t \left( \int_0^{t-r} A^{-1} S''(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) d\xi \right) \tilde{p}_j \tilde{u}(r) dr \\
&\quad + \alpha_j \int_0^t A^{-1} (S(t-r) - S(0)) (t-r)^{-\alpha_j-1} \tilde{p}_j \tilde{u}(r) dr.
\end{aligned}$$

Again by integration by parts, we find

$$\begin{aligned}
I_2(t) &= \int_0^t A^{-1} (S(t-r) - S(0)) (t-r)^{-\alpha_j} \tilde{p}_j \frac{d\tilde{u}}{dr}(r) dr \\
&= A^{-1} (S(t-r) - S(0)) (t-r)^{-\alpha_j} \tilde{p}_j \tilde{u}(r) \Big|_{r=0}^{r=t} + \int_0^t A^{-1} S'(t-r) (t-r)^{-\alpha_j} \tilde{p}_j \tilde{u}(r) dr \\
&\quad - \alpha_j \int_0^t A^{-1} (S(t-r) - S(0)) (t-r)^{-\alpha_j-1} \tilde{p}_j \tilde{u}(r) dr.
\end{aligned}$$

Now (13) yields

$$\begin{aligned}
& \|A^{-1} (S(t-r) - S(0)) (t-r)^{-\alpha_j}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} = \left\| \int_0^{t-r} A^{-1} S'(\xi) d\xi \right\|_{L^2(\Omega) \rightarrow L^2(\Omega)} (t-r)^{-\alpha_j} \\
& \leq C \int_0^{t-r} \xi^{\alpha_1-1} d\xi (t-r)^{-\alpha_j} \leq C(t-r)^{\alpha_1-\alpha_j} \rightarrow 0, \text{ as } r \rightarrow t.
\end{aligned}$$

Consequently, using  $\tilde{u}(0) = 0$ , we find

$$\begin{aligned}
& \int_0^t A^{-1} S'(t-s) (\tilde{p}_j \partial_t^{\alpha_j} \tilde{u}(s)) ds \\
&= \int_0^t A^{-1} S'(t-r) (t-r)^{-\alpha_j} \frac{\tilde{p}_j \tilde{u}(r)}{\Gamma(1-\alpha_j)} dr \\
&\quad + \int_0^t \left( \int_0^{t-r} A^{-1} S''(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) d\xi \right) \frac{\tilde{p}_j \tilde{u}(r)}{\Gamma(1-\alpha_j)} dr.
\end{aligned}$$

By Theorem 2.2 in [24], we obtain

$$\begin{aligned}
\tilde{u}(t) &= - \int_0^t A^{-1} S'(t-s) (B \cdot \nabla \tilde{u}(s) + b\tilde{u}(s) + F(s)) ds \\
&\quad + \sum_{j=2}^{\ell} \int_0^t A^{-1} S'(t-s) (t-s)^{-\alpha_j} \frac{\tilde{p}_j \tilde{u}(s)}{\Gamma(1-\alpha_j)} ds \\
&\quad + \sum_{j=2}^{\ell} \int_0^t \left( \int_0^{t-r} A^{-1} S''(t-r-\xi) (\xi^{-\alpha_j} - (t-r)^{-\alpha_j}) d\xi \right) \frac{\tilde{p}_j \tilde{u}(r)}{\Gamma(1-\alpha_j)} dr.
\end{aligned}$$

In the first and the second integrals on the right-hand side we make a change of variables  $\tau = \frac{t-s}{t}$  and in the third integral  $(\xi, r) \mapsto (\tau, \eta)$  by  $r = t - t\tau$ ,  $\xi = t\tau\eta$ , and we obtain

$$\begin{aligned}\tilde{u}(t) &= -t \int_0^1 A^{-1} S'(\tau t) (B \cdot \nabla \tilde{u}((1-\tau)t) + b\tilde{u}((1-\tau)t) + F((1-\tau)t)) d\tau \\ &\quad + \sum_{j=2}^{\ell} \frac{t^{1-\alpha_j}}{\Gamma(1-\alpha_j)} \int_0^1 A^{-1} S'(\tau t) \tau^{-\alpha_j} \tilde{p}_j \tilde{u}((1-\tau)t) d\tau \\ &\quad + \sum_{j=2}^{\ell} \frac{t^{2-\alpha_j}}{\Gamma(1-\alpha_j)} \int_0^1 \int_0^1 A^{-1} S''((1-\eta)\tau t) (\eta^{-\alpha_j} - 1) \tilde{p}_j \tilde{u}((1-\tau)t) \tau^{1-\alpha_j} d\eta d\tau.\end{aligned}\quad (15)$$

Furthermore, extending the variable  $t$  in (15) from  $(0, T)$  to the sector  $\Omega_{\theta, T}$  and setting  $\tilde{u}_0 = 0$ , we define  $\tilde{u}_{n+1}(z), n = 0, 1, \dots, z \in \Omega_{\theta, T}$  as follows:

$$\begin{aligned}\tilde{u}_{n+1}(z) &= -z \int_0^1 A^{-1} S'(\tau z) (B \cdot \nabla \tilde{u}_n((1-\tau)z) + b\tilde{u}_n((1-\tau)z) + F(\cdot, (1-\tau)z)) d\tau \\ &\quad + \sum_{j=2}^{\ell} \frac{z^{1-\alpha_j}}{\Gamma(1-\alpha_j)} \int_0^1 A^{-1} S'(\tau z) \tau^{-\alpha_j} \tilde{p}_j \tilde{u}_n((1-\tau)z) d\tau \\ &\quad + \sum_{j=2}^{\ell} \frac{z^{2-\alpha_j}}{\Gamma(1-\alpha_j)} \int_0^1 \int_0^1 A^{-1} S''((1-\eta)\tau z) (\eta^{-\alpha_j} - 1) \tilde{p}_j \tilde{u}_n((1-\tau)z) \tau^{1-\alpha_j} d\eta d\tau.\end{aligned}\quad (16)$$

By (12) - (14) we can inductively prove that  $\tilde{u}_n(z)$  is analytic in  $\Omega_{\theta}$  for any  $n \in \mathbb{N}$ .

Next we claim that the following estimates hold:

$$\|A\tilde{u}_{n+1}(z) - A\tilde{u}_n(z)\|_{L^2(\Omega)} \leq M_1 \frac{(CT^{\alpha_0} \Gamma(\alpha_0))^n}{\Gamma(n\alpha_0 + 1)}, \quad z \in \Omega_{\theta, T}, \quad n = 0, 1, 2, \dots \quad (17)$$

where

$$M_1 = T \|\lambda\|_{W^{2,\infty}(\Omega_{\theta, T})}, \quad \alpha_0 = \min_{j=2,3,\dots,\ell} \left\{ \frac{\alpha_1}{2}, \alpha_1 - \alpha_j \right\}.$$

We now prove (17) by induction on  $n$ . Firstly, for  $n = 0$ , integrating by parts and using (9) and (12), we see

$$\begin{aligned}\|A\tilde{u}_1(z) - A\tilde{u}_0(z)\|_{L^2(\Omega)} &= \|A\tilde{u}_1(z)\|_{L^2(\Omega)} = \left\| z \int_0^1 S'(\tau z) F((1-\tau)z) d\tau \right\|_{L^2(\Omega)} \\ &= \left\| S(\tau z) F(\cdot, (1-\tau)z) \Big|_{\tau=0}^{\tau=1} - \int_0^1 S(\tau z) F'(\cdot, (1-\tau)z) (-z) d\tau \right\|_{L^2(\Omega)} \\ &\leq \|S(z) F(\cdot, 0) - F(\cdot, z)\|_{L^2(\Omega)} + C \int_0^1 \|F'(\cdot, (1-\tau)z)\|_{L^2(\Omega)} d\tau \\ &\leq CT \|\lambda\|_{W^{2,\infty}(\Omega_{\theta, T})} = CM_1.\end{aligned}\quad (18)$$

Next, for any  $n \in \mathbb{N}$ , taking the operator  $A$  on both side of (16), and using (13) and (14) for the  $z \in \Omega_{\theta, T}$ , we can prove that

$$\begin{aligned}&\|A\tilde{u}_{n+1}(z) - A\tilde{u}_n(z)\|_{L^2(\Omega)} \\ &\leq C|z| \int_0^1 |\tau z|^{\frac{\alpha_1}{2}-1} \|A\tilde{u}_n((1-\tau)z) - A\tilde{u}_{n-1}((1-\tau)z)\|_{L^2(\Omega)} d\tau \\ &\quad + C \sum_{j=2}^{\ell} |z|^{1-\alpha_j} \int_0^1 |\tau z|^{\alpha_1-1} \tau^{-\alpha_j} \|A\tilde{u}_n((1-\tau)z) - A\tilde{u}_{n-1}((1-\tau)z)\|_{L^2(\Omega)} d\tau \\ &\quad + C \sum_{j=2}^{\ell} |z|^{2-\alpha_j} \int_0^1 \left( \int_0^1 ((1-\eta)\tau|z|)^{\alpha_1-2} (\eta^{-\alpha_j} - 1) d\eta \right) \tau^{1-\alpha_j} \|A\tilde{u}_n((1-\tau)z) - A\tilde{u}_{n-1}((1-\tau)z)\|_{L^2(\Omega)} d\tau.\end{aligned}$$



Here by  $B \in W^{1,\infty}(\Omega)$  and  $\|A^{\frac{1}{2}}v\|_{L^2(\Omega)} \leq C\|v\|_{H^1(\Omega)}$  and  $\|v\|_{H^2(\Omega)} \leq C\|Av\|_{L^2(\Omega)}$  for  $v \in D(A)$ , we used

$$\begin{aligned} & \|S'(\tau z)B \cdot (\nabla \tilde{u}_n - \nabla \tilde{u}_{n-1})((1-\tau)z)\|_{L^2(\Omega)} = \|A^{-\frac{1}{2}}S'(\tau z)A^{\frac{1}{2}}(B \cdot (\nabla \tilde{u}_n - \nabla \tilde{u}_{n-1})((1-\tau)z))\|_{L^2(\Omega)} \\ & \leq C\|A^{-\frac{1}{2}}S'(\tau z)\|_{L^2(\Omega) \rightarrow L^2(\Omega)}\|B \cdot (\nabla \tilde{u}_n - \nabla \tilde{u}_{n-1})((1-\tau)z)\|_{H^1(\Omega)} \\ & \leq C\|A^{-\frac{1}{2}}S'(\tau z)\|_{L^2(\Omega) \rightarrow L^2(\Omega)}\|(A\tilde{u}_n - A\tilde{u}_{n-1})((1-\tau)z)\|_{L^2(\Omega)}. \end{aligned}$$

Noting that

$$(1-\eta)^{\alpha-2} \leq \left(\frac{1}{2}\right)^{\alpha-2} \quad \text{if } \eta \in [0, \frac{1}{2}]$$

and

$$\eta^{-\alpha} - 1 \leq C(1-\eta) \quad \text{if } \eta \in [\frac{1}{2}, 1]$$

for  $0 < \alpha < 1$ , we obtain

$$\begin{aligned} \int_0^1 (1-\eta)^{\alpha_1-2}(\eta^{-\alpha_j} - 1)d\eta & \leq \int_0^{\frac{1}{2}} (1-\eta)^{\alpha_1-2}(\eta^{-\alpha_j} - 1)d\eta + \int_{\frac{1}{2}}^1 (1-\eta)^{\alpha_1-1} \frac{\eta^{-\alpha_j} - 1}{1-\eta} d\eta \\ & \leq C \int_0^{\frac{1}{2}} (\eta^{-\alpha_j} - 1)d\eta + C \int_{\frac{1}{2}}^1 (1-\eta)^{\alpha_1-1} d\eta < \infty. \end{aligned}$$

Therefore

$$\begin{aligned} & \|A\tilde{u}_{n+1}(z) - A\tilde{u}_n(z)\|_{L^2(\Omega)} \\ & \leq C \left( |z|^{\frac{\alpha_1}{2}} + \sum_{j=2}^{\ell} |z|^{\alpha_1-\alpha_j} \right) \int_0^1 (\tau^{\frac{\alpha_1}{2}-1} + \sum_{j=2}^{\ell} \tau^{\alpha_1-\alpha_j-1}) \|A\tilde{u}_n((1-\tau)z) - A\tilde{u}_{n-1}((1-\tau)z)\|_{L^2(\Omega)} d\tau \\ & \leq C|z|^{\alpha_0} \int_0^1 \tau^{\alpha_0-1} \|A\tilde{u}_n((1-\tau)z) - A\tilde{u}_{n-1}((1-\tau)z)\|_{L^2(\Omega)} d\tau, \quad z \in \Omega_{\theta,T}. \end{aligned}$$

Therefore

$$\|A\tilde{u}_{n+1}(z) - A\tilde{u}_n(z)\|_{L^2(\Omega)} \leq C|z|^{\alpha_0} \int_0^1 \tau^{\alpha_0-1} \|A\tilde{u}_n((1-\tau)z) - A\tilde{u}_{n-1}((1-\tau)z)\|_{L^2(\Omega)} d\tau \quad (19)$$

for  $z \in \Omega_{\theta,T}$ . We note that  $\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$  and  $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$  for  $\alpha, \beta > 0$ . Iterating (19), in terms of (18), we obtain

$$\begin{aligned} \|A\tilde{u}_2(z) - A\tilde{u}_1(z)\|_{L^2(\Omega)} & \leq C|z|^{\alpha_0} \int_0^1 \tau^{\alpha_0-1} M_1 d\tau = \frac{CM_1}{\alpha_0} |z|^{\alpha_0}, \\ \|A\tilde{u}_3(z) - A\tilde{u}_2(z)\|_{L^2(\Omega)} & \leq C|z|^{\alpha_0} \int_0^1 \tau^{\alpha_0-1} \frac{CM_1}{\alpha_0} |(1-\tau)z|^{\alpha_0} d\tau \\ & = \frac{(C|z|^{\alpha_0})^2 M_1}{\alpha_0} \frac{\Gamma(\alpha_0)\Gamma(\alpha_0+1)}{\Gamma(2\alpha_0+1)} = \frac{(C|z|^{\alpha_0}\Gamma(\alpha_0))^2 M_1}{\Gamma(2\alpha_0+1)}, \end{aligned}$$

and

$$\|A\tilde{u}_4(z) - A\tilde{u}_3(z)\|_{L^2(\Omega)} \leq C|z|^{\alpha_0} \int_0^1 \tau^{\alpha_0-1} \frac{M_1(C|(1-\tau)z|^{\alpha_0}\Gamma(\alpha_0))^2}{\Gamma(2\alpha_0+1)} d\tau = \frac{(C|z|^{\alpha_0}\Gamma(\alpha_0))^3 M_1}{\Gamma(3\alpha_0+1)}, \quad \text{etc.}$$

Therefore similarly we obtain

$$\|A\tilde{u}_{n+1}(z) - A\tilde{u}_n(z)\|_{L^2(\Omega)} \leq \frac{(C|z|^{\alpha_0}\Gamma(\alpha_0))^n}{\Gamma(n\alpha_0+1)} M_1 \leq \frac{(CT^{\alpha_0}\Gamma(\alpha_0))^n}{\Gamma(n\alpha_0+1)} M_1, \quad n = 0, 1, 2, \dots, \forall z \in \Omega_{\theta,T}.$$

Using (11), we see that

$$\sum_{n=0}^{\infty} \frac{(CT^{\alpha_0}\Gamma(\alpha_0))^n}{\Gamma(n\alpha_0 + 1)} < \infty.$$

Hence the majorant test implies  $\sum_{n=0}^{\infty} \|A\tilde{u}_{n+1}(z) - A\tilde{u}_n(z)\|_{L^2(\Omega)}$  is convergent uniformly in  $z \in \Omega_{\theta,T}$ . Therefore there exists  $Au_*(z) \in L^2(\Omega)$  such that  $\|A\tilde{u}_n(z) - Au_*(z)\|_{L^2(\Omega)}$  tends to 0 as  $n \rightarrow \infty$  uniformly in  $z \in \Omega_{\theta,T}$ . Therefore  $Au_*(z)$  is analytic in  $\Omega_{\theta,T}$ . Moreover, since  $T$  is arbitrarily chosen, we deduce  $Au_*(z)$  is analytic in the sector  $\Omega_{\theta}$ .

Next we prove (5). In view of  $p \leq 0$  on  $\overline{\Omega}$ , we can prove

$$u(x, t) \leq \max\{0, \max_{x \in \partial\Omega, 0 \leq t \leq T} g(x)\lambda(t)\} \quad \text{for } x \in \overline{\Omega}, 0 \leq t \leq T. \quad (20)$$

In fact, we can repeat the proof of Theorem 2 in Luchko [17] which assumes that  $p_1, \dots, p_{\ell}$  are all constants and  $p_1 > 0$ ,  $p_j \geq 0$  for  $j = 2, \dots, \ell$ . Therefore (20) holds if  $u$  is sufficiently smooth. For our solution with the boundary value  $g(x)\lambda(t)$ , applying an approximating argument similar to Theorems 4 and 5 in [23], we see (20) for the solutions constructed in the theorem.

Replacing  $u$  by  $-u$  and applying (20), we obtain

$$-u(x, t) \leq \max\{0, \max_{x \in \partial\Omega, 0 \leq t \leq T} (-g(x)\lambda(t))\},$$

that is,

$$u(x, t) \geq \min\{0, \min_{x \in \partial\Omega, 0 \leq t \leq T} g(x)\lambda(t)\}$$

for  $x \in \overline{\Omega}$  and  $0 \leq t \leq T$ . With (20), we obtain

$$|u(x, t)| \leq \max_{x \in \partial\Omega, 0 \leq t \leq T} |g(x)\lambda(t)|$$

for  $x \in \overline{\Omega}$  and  $0 \leq t \leq T$ . Therefore the proof of (5) is completed.

Finally we show that  $u_*(z)$  is the mild solution  $\tilde{u}$  to (6) when the variable  $z$  is restricted to  $(0, T)$ . In fact, denoting the imaginary part of  $u_*(t)$ ,  $\forall t \in (0, T)$  as  $\text{Im } u_*(t)$ , we see that  $v = \text{Re } u_*(t)$  is a mild solution to the following initial-boundary problem:

$$\begin{cases} \partial_t^{\alpha_1} v + \sum_{j=2}^{\ell} p_j(x) \partial_t^{\alpha_j} v = \text{div}(\frac{1}{p_1(x)} \nabla v) + B(x) \cdot \nabla v + b(x)v, & (x, t) \in \Omega \times (0, T), \\ v(x, 0) = 0, & x \in \Omega, \\ v(x, t) = 0, & x \in \partial\Omega, t \in (0, T). \end{cases}$$

Using the uniqueness result of the above problem (e.g., Theorem 2.4 in [12]), we have  $\text{Im } u_*(t) = 0, \forall t \in (0, T)$ . Thus again by the uniqueness argument we see that  $u_*(t) = u(t), \forall t \in (0, T)$ . Consequently, we see that  $\tilde{u}(t) = u(t) - \lambda(t)\tilde{g}$  is analytic from  $[0, T]$  to  $H^2(\Omega)$  in view of the analyticity of  $\lambda(t)$ . This completes the proof of the theorem.  $\square$

### 3 Uniqueness for inverse boundary value problem

The proof of Corollary 1.1 is exactly the same as the proof of Theorem 1.1, and the only difference is that instead of the uniqueness result of [26], we have to use the uniqueness result in [8]. Thus it is sufficient to prove Theorem 1.1

*Proof of Theorem 1.1.* We reduce the inverse problem to the corresponding inverse boundary value problem for the Schrödinger equation

$$\begin{cases} \Delta v(x, s) + P_s(x)v(x, s) = 0, & x \in \Omega, \\ v(x, s) = g(x), & x \in \partial\Omega, \end{cases}$$

for all large  $s > 0$ . Here and henceforth we set

$$P_s(x) := p(x) - \sum_{j=1}^{\ell} p_j(x)s^{\alpha_j}, \quad Q_s(x) := q(x) - \sum_{j=1}^m q_j(x)s^{\beta_j}.$$

Let  $u_1(g)(x, t)$  and  $u_2(g)(x, t)$  be the solutions to (1) with  $(\ell, \vec{\alpha}, p_j, p)$  and  $(m, \vec{\beta}, q_j, q)$  respectively. Since  $\lambda(t)$  is  $t$ -analytic in  $t > 0$ , Theorem 2.1 implies that  $\Delta u_1(g)(x, t)$  and  $\Delta u_2(g)(x, t)$  are  $t$ -analytic in  $t > 0$  for any fixed  $x \in \overline{\Omega}$ . Therefore, since  $w \mapsto \frac{\partial w}{\partial \nu}: H^{\frac{3}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$  is continuous, equality (4) implies

$$\frac{\partial u_1(g)}{\partial \nu}(x, t) = \frac{\partial u_2(g)}{\partial \nu}(x, t), \quad x \in \partial\Omega, 0 < t < \infty \quad \text{for } g \in C^\infty(\partial\Omega).$$

Let  $(Lu)(x, s) := \int_0^\infty e^{-st}u(x, t)dt$  be the Laplace transform of  $u(x, t)$  in  $t$  for each fixed  $x \in \overline{\Omega}$ . By (5) in Theorem 2.1 and assumption  $|\lambda(t)| \leq C_0 e^{C_0 t}$  for  $t > 0$ , we see that  $|u(x, t)| \leq C e^{C_0 t}$  for  $t > 0$ , where  $C > 0$  is a constant and is independent of  $t > 0$  and  $x \in \Omega$ . Therefore  $(Lu_k(g))(x, s)$ ,  $k = 1, 2$ , exist for  $s > C_1$  where  $C_1 > 0$  is some constant depending only on  $\lambda$ . Using  $u_k(g)(x, 0) = 0$ , by [23], we have

$$L(\partial_t^\alpha u_k(g))(x, s) = s^\alpha (Lu_k(g))(x, s), \quad s > C_1, k = 1, 2.$$

Therefore by the fractional diffusion equations themselves, it follows that  $L(\Delta u_k(g))(x, s)$ ,  $k = 1, 2$ , exist for  $s > C_1$ . Hence

$$\begin{cases} \Delta L(u_1(g))(x, s) + P_s(x)L(u_1(g))(x, s) = 0, & x \in \Omega, s > C_1, \\ L(u_1(g))(x, s) = (L\lambda)(s)g(x), & x \in \partial\Omega, s > C_1, \end{cases}$$

$$\begin{cases} \Delta L(u_2(g))(x, s) + Q_s(x)L(u_2(g))(x, s) = 0, & x \in \Omega, s > C_1, \\ L(u_2(g))(x, s) = (L\lambda)(s)g(x), & \forall x \in \partial\Omega, \text{ and } \forall s > C_1, \end{cases}$$

and

$$\frac{\partial L(u_1(g))}{\partial \nu}(x, s) = \frac{\partial L(u_2(g))}{\partial \nu}(x, s), \quad \forall x \in \partial\Omega, \text{ and } \forall s > C_1.$$

Next we consider the following two boundary value problems

$$\begin{cases} \Delta v_1(x, s) + P_s(x)v_1(x, s) = 0, & x \in \Omega, s > C_1, \\ v_1(x, s) = g(x), & \forall x \in \partial\Omega, \text{ and } \forall s > C_1. \end{cases} \quad (21)$$

and

$$\begin{cases} \Delta v_2(x, s) + Q_s(x)v_2(x, s) = 0, & x \in \Omega, s > C_1, \\ v_2(x, s) = g(x), & \forall x \in \partial\Omega, \text{ and } \forall s > C_1. \end{cases} \quad (22)$$

Then we define their Dirichlet-to-Neumann maps  $\Lambda(P_s)$  and  $\Lambda(Q_s)$  by

$$\Lambda(P_s)g := \frac{\partial v_1(g)}{\partial \nu}|_{\partial\Omega}, \quad \Lambda(Q_s)g := \frac{\partial v_2(g)}{\partial \nu}|_{\partial\Omega}.$$

Now we prove that there exists a subset  $\sigma \subset (C_1, \infty)$  such that  $\sigma$  contains a non-empty open interval and

$$\Lambda(\ell, \vec{\alpha}, p_j, p)g = \Lambda(m, \vec{\beta}, q_j, q)g \implies \Lambda(P_s)g = \Lambda(Q_s)g \quad \text{for all } g \in C^\infty(\partial\Omega) \text{ and } s \in \sigma. \quad (23)$$

In fact,  $(L\lambda)(z)$  is analytic in  $\text{Re } z > C_1$  and  $\{s; (L\lambda)(s) = 0, s > C_1\}$  has no accumulation points except for  $\infty$ . Therefore  $\sigma := (C_1, \infty) \setminus \{s; (L\lambda)(s) = 0, s > C_1\}$  contains a non-empty open interval. Then we can set  $\tilde{v}_j(g)(x, s) = \frac{L(u_j(g))(x, s)}{(L\lambda)(s)}$  for  $j = 1, 2$  and  $s \in \sigma$ . It is not very difficult to see that  $\tilde{v}_1(g)$  and  $\tilde{v}_2(g)$  are the solutions to (21) and (22) respectively. From the uniqueness of the boundary value problem, we see that  $\tilde{v}_j(g) = v_j(g)$ ,  $j = 1, 2$  for  $s \in \sigma$ .

Here by the density of  $C^\infty(\partial\Omega)$  in  $H^{\frac{1}{2}}(\partial\Omega)$  and the continuity of  $\Lambda(P_s) : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ , we see that (23) holds for all  $g \in H^{\frac{1}{2}}(\partial\Omega)$ .

Therefore the uniqueness [26] by Dirichlet-to-Neumann map in determining a potential, we see that  $P_s(x) = Q_s(x)$  for all  $x \in \Omega$  and  $s \in \sigma$ . Since  $\sigma$  contains a non-empty open interval, we obtain  $\ell = m$ ,  $\vec{\alpha} = \vec{\beta}$ ,  $p_j = q_j$ ,  $1 \leq j \leq \ell$  and  $p = q$ . Thus the proof of the theorem is completed.  $\square$

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